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UNSTEADY HEAT TRANSFER IN LAMINAR BOUNDARY LAYER OVER A FLAT PLATE

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Abstract—The thermal response of an incompressible, laminar boundary layer over a semi-infinite flat plate due to a step change in either the wall temperature or wall flux is investigated. A new analytical method of finding solutions valid for all times is presented. Results for wall flux or wall temperature transients are displayed graphically for Prandtl numbers ranging from 0·01 to 100. It is demonstrated that, for a restricted range of Prandtl number (0·72 to 100), the time required for the thermal layer to achieve steadiness varies inversely with the free stream velocity and directly with $\frac{1}{3}$ power of the Prandtl number.

For an ideal gas of constant Prandtl number with linear viscosity–temperature relation of the Chapman–Rubesin description, the incompressible results for both cases can be directly applied when the fluid properties are properly interpreted.

NOMENCLATURE

c_p, specific heat at constant pressure;
k, thermal conductivity;
Nu, Nusselt number, $= q_w x/(T_w - T_{aw})\,k$; for compressible flow with a step change in surface temperature, k is to be evaluated at T_w;
p, parameter in Laplace transform;
Pr, Prandtl number, $= c_p\mu/k$;
q, heat flux;
Re, Reynolds number, $= u_\infty x/\nu$; for compressible flow with a step change in surface temperature, ν is to be evaluated at T_w;
T, temperature;
t, time;
u, velocity component in x-direction;
v, velocity component in y-direction;
x, coordinate along the plate;
y, coordinate normal to the plate;
$1(t)$, Heaviside unit operator, $= 0$ for $t < 0$ and $= 1$ for $t \geqslant 0$.

Greek symbols
κ, thermal diffusivity;
μ, dynamic viscosity;
ν, kinematic viscosity;
ρ, density.

Subscripts
s, steady state;
w, wall;
aw, adiabatic wall;
∞, free stream condition.

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1. INTRODUCTION

THE THERMAL response behavior of an incompressible, steady laminar boundary layer over a flat plate due to a step change in surface temperature has been studied by Cess [1]. Series solutions corresponding to small and large times were deduced in the Laplace transform plane and were joined by an essentially empirical curve fitting procedure devised by Rosenzweig [2]. Possible pitfalls of Rozenzweig's method has been pointed out [3]. In a second paper on the same subject, Cess [4] restricted his attention to low Prandtl number fluids and, hence, argued that the velocity components in the energy equation could be approximated by their values in the outer regions of the velocity boundary layer. This contention of Cess was disputed by Riley [5] who showed that, for both the initial growth of the thermal layer and its eventual decay to the steady state, the velocity components near the wall should be used regardless of the Prandtl number of the fluid. However, the solution given by Riley for large times contains an unspecified constant and, thus, is incomplete. In this paper, a totally different approach is followed. The energy equation in the Laplace transform variable was further transformed to a suitable form for which a series solution, valid for all times, could be deduced. Results for wall flux variation were obtained for Prandtl numbers ranging from 0·01 to 100. The same technique was then applied to examine the case in which the wall heat flux underwent a step change. To the authors' knowledge, solution for the latter problem has not been previously reported in the literature.

2. GOVERNING EQUATIONS

The physical model and the coordinate system are illustrated in Fig. 1. The objective of the

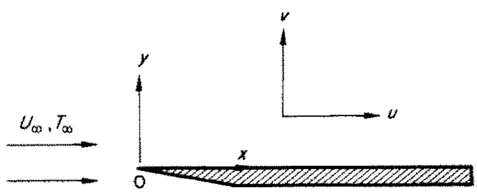

FIG. 1. Physical model and coordinate system.

analysis is to ascertain the entire time-history of the heat-transfer process due to either a step change in plate surface temperature or heat flux. The flow is steady. The conservation equations for the boundary-layer flow are well known. For constant properties, they are:

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

energy:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{3}$$

The boundary conditions for the velocity field are

$$u(x, 0) = v(x, 0) = 0; \qquad u(x, \infty) = u_\infty. \tag{4}$$

The initial condition for the temperature field is

$$T(x, y, 0) = T_\infty + T_i(x, y) \tag{5}$$

and the boundary conditions are

$$T(x, 0, t) = T_{aw} + (T_w - T_{aw})\,1(t); \qquad T(x, \infty, t) = T_\infty \tag{6a}$$

for the case of a step change in surface temperature and

$$\frac{\partial T}{\partial y}(x, 0, t) = -\frac{q_w}{k}\,1(t); \qquad T(x, \infty, t) = T_\infty \tag{6b}$$

for the case of a step change in surface flux. In (5), T_i designates the temperature field due to viscous dissipation prior to the disturbance and is given by Pohlhausen's plate thermometer solution [6]. At the plate surface, it becomes the adiabatic wall temperature in excess of the free stream temperature, $T_{aw} - T_\infty$.

The solution of (1) and (2) satisfying (4) is well known and was first given by Blasius:

$$u = u_\infty f'(\eta), \qquad v = \left(\frac{u_\infty \nu}{2x}\right)^{\frac{1}{2}}(\eta f' - f) \tag{7}$$

where η is the similarity variable defined by

$$\eta = y\left(\frac{u_\infty}{2\nu x}\right)^{\frac{1}{2}} \tag{8}$$

and f is the power series

$$f = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\eta^n \tag{9}$$

with $f(0) = f^{(1)}(0) = 0$, $f^{(2)}(0) = a$, $f^{(3)}(0) = f^{(4)}(0) = 0, f^{(5)}(0) = -a^2, f^{(6)}(0) = f^{(7)}(0) = 0$, $f^{(8)}(0) = 11a^3$, etc. and $a = 0{\cdot}4696$ as calculated by Hartree [7]. In (7) and others which follow, the prime denotes differentiation with respect to η.

Dimensionless form of energy equation

To facilitate analysis and discussion, we introduce the following dimensionless quantities

$$\tau = \frac{u_\infty t}{x} \tag{10}$$

$$\theta = \frac{T - T_\infty}{T_w - T_{aw}} - \frac{u_\infty^2}{2c_p(T_w - T_{aw})}\Phi(\eta) \tag{11a}$$

for a step change in wall temperature and

$$\theta = \frac{T - T_\infty}{\frac{q_w}{k}\left(\frac{2\nu x}{u_\infty}\right)^{\frac{1}{2}}} - \frac{u_\infty^2}{2c_p\frac{q_w}{k}\left(\frac{2\nu x}{u_\infty}\right)^{\frac{1}{2}}}\Phi(\eta) \tag{11b}$$

for a step change in wall flux. It can be shown that, for the former, the energy equation transforms to

$$2\,Pr(1 - f'\tau)\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\eta^2} + Pr\,f\frac{\partial\theta}{\partial\eta} \tag{12a}$$

with

$$\theta(\eta, 0) = 0; \qquad \theta(0, \tau) = 1(\tau), \qquad \theta(\infty, \tau) = 0 \tag{13a}$$

and, for the latter, it becomes

$$2\,Pr(1 - f'\tau)\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\eta^2} + Pr\,f\frac{\partial\theta}{\partial\eta} - Pr\,f'\theta \tag{12b}$$

with

$$\theta(\eta, 0) = 0; \qquad \frac{\partial\theta}{\partial\eta}(0, \tau) = -1(t), \qquad \theta(\infty, \tau) = 0. \tag{13b}$$

In either case, $\Phi(\eta)$ satisfies the ordinary differential equation:

$$\Phi'' + Pr\,f\Phi' = -2\,Pr(f'')^2 \tag{14}$$

with

$$\Phi'(0) = 0, \qquad \Phi(\infty) = 0. \tag{15}$$

Equation (14) with boundary conditions (15) has been integrated by Pohlhausen who presented the results in graphical form for Pr ranging from 0·6 to 15.

3. SOLUTION METHOD AND RESULTS

(A) *Step change in wall temperature*

We define the Laplace transform of θ in the usual way, i.e.

$$\bar\theta(\eta; p) = \int_0^\infty e^{-p\tau}\theta(\eta, \tau)\,d\tau \tag{16}$$

and obtain from (12a) and (13a)

$$\bar\theta'' + Pr\,f\bar\theta' = 2\,Pr\left[f'\frac{\partial(p\bar\theta)}{\partial p} + p\bar\theta\right] \tag{17}$$

with

$$\bar\theta(0) = p^{-1}, \qquad \bar\theta(\infty) = 0 \tag{18}$$

wherein the parametric dependence of $\bar\theta$ on p is understood. To further transform (17) into a form suitable for series solution, we introduce a new dependent variable $\bar Y$ defined by

$$\bar Y = p\bar\theta\exp\left[Pr\frac{F(\eta)}{2}\right] \tag{19}$$

where

$$F(\eta) = \int_0^\eta f\,d\eta.$$

It follows then

$$\bar Y'' - 2\,Pr\left(p + \frac{f'}{4} + \frac{Pr}{8}f^2\right)\bar Y = 2\,Pr\,f'p\frac{\partial\bar Y}{\partial p} \tag{20}$$

with

$$\bar Y(0) = 1, \qquad \bar Y(\infty) = 0. \tag{21}$$

We seek a series solution of the form

$$\bar Y = \exp\{-[2\,Pr(p + \lambda)]^{\frac{1}{2}}\eta\} \times \sum_{n=0}^\infty u_n(\eta)[2\,Pr(p + \lambda)]^{-n/2} \tag{22}$$

which satisfies $\bar Y(\infty) = 0$ for $(p + \lambda) > 0$ and all u_n's bounded. Furthermore, we set $u_0(0) = 1$, and $u_1(0) = u_2(0) = \ldots = u_n(0) = 0$; hence $\bar{Y}(0) = 1$. The unknown, real, positive constant λ† is to be evaluated from the steady state solution of the problem as will be explained later. The form of the solution chosen above is suggested by, but modified from, Langer's technique‡ of finding asymptotic solutions of second order differential equations with respect to a large parameter. The artifice of introducing the unknown parameter λ renders the series (22) suitable for all times.

Upon substituting (22) into (20) and equating the coefficients of like powers of $(p + \lambda)$, we find

$$u_0' = \frac{Pr}{2}\eta f' u_0 \tag{23}$$

and

$$u_n' = \frac{u_{n-1}''}{2} - \frac{Pr}{4}\left[(3 - 2n)f' + \frac{Pr}{2}f^2\right]u_{n-1} + \frac{Pr}{2}\eta f' u_n + \lambda\, Pr\,[u_{n-1} - Pr\,\eta f' u_{n-2} - (n - 3)\,Pr\,f' u_{n-3}], \quad \text{for } n \geqslant 1 \text{ and } u_{-1} \equiv u_{-2} \equiv 0. \tag{24}$$

Through successive differentiation of (23) and (24), followed by inserting values of $f^{(n)}(0)$, we obtain

$$\left.\begin{aligned}
&u_0'(0) = 0, \quad u_1'(0) = \lambda\, Pr, \quad u_2'(0) = \frac{1}{8}a\, Pr, \quad u_3'(0) = \frac{1}{2}\lambda^2 Pr^2,\\
&u_4'(0) = \frac{1}{4}a\lambda\, Pr^2, \quad u_5'(0) = -\frac{1}{16}a^2 Pr + \frac{53}{128}a^2 Pr^2 + \frac{1}{2}\lambda^3 Pr^3,\\
&u_6'(0) = \frac{1}{2}a\lambda^2 Pr^3, \quad u_7'(0) = -\frac{5}{16}a^2\lambda\, Pr^2 + \frac{265}{128}a^2\lambda\, Pr^3 + \frac{5}{8}\lambda^4 Pr^4,\\
&u_8'(0) = \frac{77}{512}a^3 Pr - \frac{809}{512}a^3 Pr^2 + \frac{2625}{512}a^3 Pr^3 + a\lambda^3 Pr^4, \text{ etc.}
\end{aligned}\right\} \tag{25}$$

From (19) and (22), we readily find

$$-\bar{\theta}'(0) = p^{-1}[2\,Pr(p + \lambda)]^{\frac{1}{2}} - p^{-1}\sum_{n=0}^{\infty} u_n'(0)[2\,Pr(p + \lambda)]^{-n/2}. \tag{26}$$

The inverse of (26) is

$$-\theta'(0, \tau) = (2\,Pr)^{\frac{1}{2}}\left[\frac{\exp(-\lambda\tau)}{(\pi\tau)^{\frac{1}{2}}} + \lambda^{\frac{1}{2}}\,\mathrm{erf}\,(\lambda\tau)^{\frac{1}{2}}\right] - \sum_{n=0}^{\infty}\frac{u_n'(0)}{(2\lambda\, Pr)^{n/2}}\frac{\Gamma_{\lambda\tau}(n/2)}{\Gamma(n/2)} \tag{27}$$

where erf, $\Gamma_{\lambda\tau}$ and Γ denote respectively the error, incomplete gamma and complete gamma functions. As $\tau \to \infty$, (27) reduces to

$$-\theta_s'(0) = (2\lambda\, Pr)^{\frac{1}{2}} - \sum_{n=0}^{\infty}\frac{u_n'(0)}{(2\lambda\, Pr)^{n/2}}. \tag{28}$$

To evaluate λ, we need to determine separately the steady-state solution. This has been reported in the literature and is briefly described below.

(A.1) *Steady-state solution and evaluation of λ*

If we denote the steady temperatures by $\theta_s(\eta)$, it is clear from (12a) and (13a) that

$$\theta_s'' + Pr\, f\theta_s' = 0 \tag{29}$$

and

$$\theta_s(0) = 1, \quad \theta_s(\infty) = 0. \tag{30}$$

A procedure of integrating (29) and some

† In this problem, λ is real. It is conceivable that, in general, λ may be complex; then $Re(\lambda) > 0$.

‡ Chapter 19 of [8] gives a concise presentation of the technique; it also contains a list of Langer's original work on the subject.

numerical results for Pr ranging from 0·6 to 15 were first given in [6]. Meksyn [8] developed an asymptotic method of integration and showed that

$$-\theta_s'(0) = \left[\int_0^\infty \exp(-Pr\,F)\,d\eta\right]^{-1} = \frac{0{\cdot}4790\,Pr^{\frac{1}{3}}}{C(Pr)} \quad (31)$$

where

$$C(Pr) = 1 + 2{\cdot}222 \times 10^{-2}\,Pr^{-1} - 2{\cdot}469 \times 10^{-3}\,Pr^{-2} + 2{\cdot}677 \times 10^{-4}\,Pr^{-3} + 2{\cdot}473 \times 10^{-4}\,Pr^{-4} - 1{\cdot}189 \times 10^{-4}\,Pr^{-5} + \ldots \quad (32)$$

The numerical constant 0·4790 and others in (32) were recalculated by us. They differ slightly from those originally given by Meksyn. For moderate and large Pr, CPr is a slowly varying function as can be seen from Table 1. Included in that table are values of $-\theta_s'(0)$ calculated from (31). For $Pr = 0{\cdot}1$ and 0·01, the series becomes semi-divergent and use was made of Euler's procedure for the evaluation of the sum.

With $\theta_s'(0)$ known, λ can be determined from (28) for the corresponding Prandtl number. An iterative computer routine was used for this purpose and the results are as shown. We note that the series on the right-hand side of (28) is semi-divergent; its sum was calculated by applying Euler's procedure.

Table 1. Values of $C(Pr)$, $-\theta_s'(0)$ and λ

Pr	0·01	0·01	0·72	1	10	100
$C(Pr)$	1·389	1·156	1·027	1·020	1·002	1·000
$-\theta_s'(0)$	$7{\cdot}430 \times 10^{-2}$	$1{\cdot}924 \times 10^{-1}$	$4{\cdot}181 \times 10^{-1}$	$4{\cdot}696 \times 10^{-1}$	1·030	2·223
λ	6·372	2·189	1·403	1·272	0·6070	0·2825

(A.2) *Heat-transfer results*

The local heat flux at the plate surface is

$$q_w = -k(T_w - T_{aw})\left(\frac{u_\infty}{2\nu x}\right)^{\frac{1}{2}} \theta'(0,\tau) \quad (33)$$

and the corresponding Nusselt number is

$$Nu = \frac{q_w x}{(T_w - T_{aw})\,k} = Re^{\frac{1}{2}}\,\frac{-\theta'(0,\tau)}{\sqrt{2}}. \quad (34)$$

Accordingly,

$$Nu\,Re^{-\frac{1}{2}} = Pr^{\frac{1}{2}}\left[\frac{\exp(-\lambda\tau)}{\sqrt{(\pi\tau)}} + \lambda^{\frac{1}{2}}\,\text{erf}\,(\lambda\tau)^{\frac{1}{2}}\right] - \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{u_n'(0)}{(2\lambda\,Pr)^{n/2}}\,\frac{\Gamma_{\lambda\tau}(n/2)}{\Gamma(n/2)}. \quad (35)$$

(a) *Solution useful for small time.* For small $\lambda\tau$, we use the following series expansion for $\exp(-x)$, $\text{erf}\,x$ and $\Gamma_x(n/2)$.

$$\exp(-x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} \quad (36.1)$$

$$\text{erf}\,x = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} \quad (36.2)$$

$$\Gamma_x\left(\frac{n}{2}\right) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{n/2+m}}{(n/2+m)\,m!}. \quad (36.3)$$

Upon substituting the foregoing into (35), combining terms and simplifying, we find

$$Nu\,Re^{-\frac{1}{2}} = \frac{Pr^{\frac{1}{2}}}{\sqrt{(\pi\tau)}} - \frac{a}{16\sqrt{2}}\tau + \frac{a^2}{240\sqrt{\pi}}Pr^{-\frac{1}{2}} \times \left(1 - \frac{53}{8}Pr\right)\tau^{\frac{3}{2}} - \frac{a^3}{196608\sqrt{2}}Pr^{-3} \times (77 - 809\,Pr + 2625\,Pr^2)\,\tau^4 + \ldots. \quad (37)$$

It is interesting to note that λ does not appear in (37). The first term represents the conduction transient as expected. If the numerical value of a (=0·4696) is inserted in (37), the result is

$$Nu\, Re^{-\frac{1}{2}} = 0{\cdot}5642\, Pr^{\frac{1}{2}}\, \tau^{-\frac{1}{2}} - 0{\cdot}2075 \times 10^{-1}\tau - 0{\cdot}518 \times 10^{-3}\, Pr^{-\frac{3}{2}} \times (6{\cdot}63\, Pr - 1)\, \tau^{\frac{5}{2}} + 0{\cdot}287 \times 10^{-4}\, Pr^{-3} \times (10{\cdot}5\, Pr - 1 - 34{\cdot}1\, Pr^2)\, \tau^4 + \ldots . \quad (38)$$

The first two terms agree with the corresponding terms of a series given by Riley [5]. The third term differs in the coefficient of Pr for which Riley reported the value 4·9 instead of 6·63. This minor discrepancy is not unexpected since Riley employed only two terms of the power series for the Blasius function f. The fourth term was not reported in [5]. With reference to his method of solution, Riley remarked:

> Further terms of the series may be obtained by retaining more terms of the series for f..., (but) at each stage the details become more tedious.

Hence, not only have we succeeded in obtaining a solution valid for all times but also the arithmetic involved in our procedure is simpler even though more terms are retained in the Blasius series.

(b) *Solution useful for large time.* For large $\lambda\tau$, we employ the following asymptotic series,

$$\operatorname{erf} x \simeq 1 - \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{2^2 x^4} - \ldots\right) \quad (39.1)$$

$$\frac{\Gamma_x(n/2)}{\Gamma(n/2)} \simeq 1 - \frac{x^{n/2-1}\, e^{-x}}{\Gamma(n/2)}\left[1 + \left(\frac{n}{2} - 1\right)\frac{1}{x} + \left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)\frac{1}{x^2} + \ldots\right]. \quad (39.2)$$

For any finite x and n, $x^{n/2-1}\, e^{-x}/\Gamma(n/2)$ is always less than unity, it becomes zero when $x \to \infty$.

Substituting (39.1) and (39.2) into (35) and separating out the time independent terms, we find

$$Nu\, Re^{-\frac{1}{2}} \simeq (\lambda\, Pr)^{\frac{1}{2}} - \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{u_n'(0)}{(2\lambda\, Pr)^{n/2}} + \frac{(\lambda\, Pr)^{\frac{1}{2}}}{2\sqrt{\pi}}\, e^{-\lambda\tau}(\lambda\tau)^{-\frac{3}{2}}\left(1 - \frac{3}{2}\frac{1}{\lambda\tau} + \ldots\right) + \frac{e^{-\lambda\tau}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{u_n'(0)}{(2\lambda\, Pr)^{n/2}}\, \frac{(\lambda\tau)^{n/2-1}}{\Gamma(n/2)}\left[1 + \left(\frac{n}{2} - 1\right)\frac{1}{\lambda\tau} + \left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)\frac{1}{(\lambda\tau)^2} + \ldots\right] \quad (40.1)$$

$$= -\frac{\theta_s'(0)}{\sqrt{2}} + \Delta \quad (40.2)$$

where Δ denotes the sum of all the terms in (40.1) containing $e^{-\lambda\tau}$. It represents the deviation from the steady state. The series involved in Δ are asymptotic and should be used as such. The precise manner by which the wall flux approaches the steady state during the final stage of decay seems quite complicated.

For engineering applications, one is often concerned with the ratio of the instantaneous wall flux to its steady value. Thus,

$$\frac{q_w}{q_{w,s}} = \frac{Nu}{Nu_s} = \frac{\theta'(0, \tau)}{\theta_s'(0)} \quad (41)$$

wherein $\theta'(0, \tau)$ and $\theta_s'(0)$ are respectively given by (27) and (31). Figure 2 displays graphically this ratio on log–log coordinates for Pr ranging from 0·01 to 100. The long time results for $Pr = 0{\cdot}01$ should be used with some reservation since the boundary-layer approximation which is implicit in the governing energy equation may become poor.

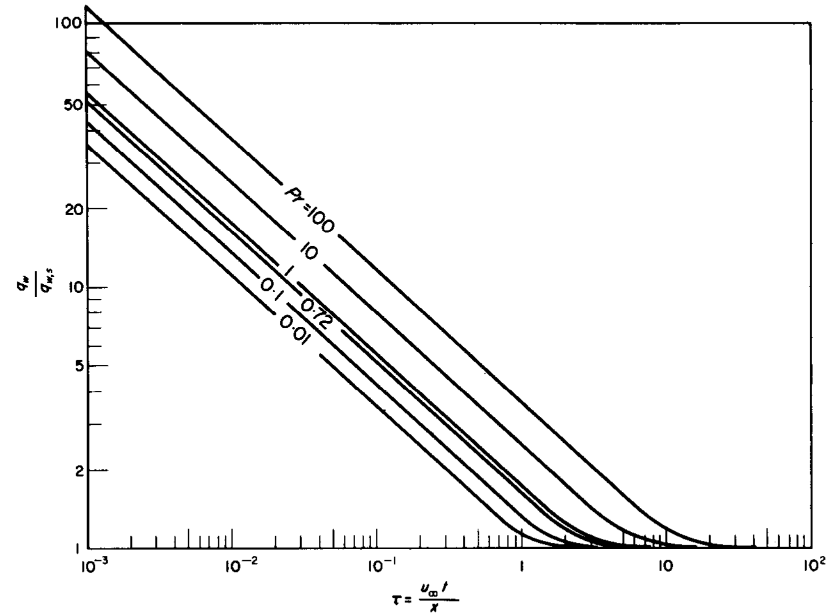

FIG. 2. Wall heat flux response of a flat plate due to a step change in surface temperature.

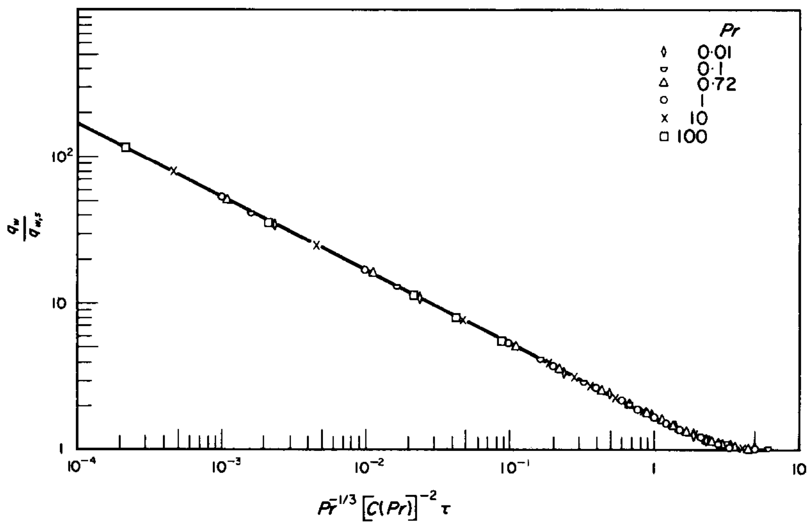

FIG. 3. Correlation of all data used in plotting Fig. 2.

If one replots the wall flux ratio $q_w/q_{w,s}$ against $Pr^{-\frac{1}{3}}[C(Pr)]^{-2}\,\tau$, the data could, for all practical purposes, be brought to lie on a single curve for the range of Pr investigated. This is illustrated in Fig. 3. As noted earlier, $C(Pr)$ is close to unity when Pr equals or exceeds 0·72. For such fluids, the response time of the thermal layer due to a step change in plate temperature increases monotonically with increasing Prandtl number in accordance with $Pr^{\frac{1}{3}}$.

Inasmuch as the energy equation is linear, the foregoing results can be readily generalized for any arbitrary but uniform surface temperature variation with time, $T_w(t)$ for $t > 0$. From the well known Duhamel's theorem, the wall flux in this case may be conveniently expressed by the Stieltjes integral

$$q_w = -k\left(\frac{u_\infty}{2\nu x}\right)^{\frac{1}{2}} \int_0^\tau \theta'(0, \tau - t^*)\, \mathrm{d}T_w(t^*) \qquad (42)$$

where t^* is a dummy variable of integration. Thus far, we have limited our consideration to the condition $T_w(0) = T_{aw}$. Further generalization to include a constant temperature difference $T_w - T_{aw}$ for $t < 0$ is obvious.

(B) *Step change in wall flux*

We now seek the solution of (12b) satisfying (13b). As in the previous case, the Laplace transform was first applied, followed by introducing the dependent variable $\bar{Y}$ defined by (19). It has been found that $\bar{Y}$ satisfies

$$\bar{Y}'' - 2\,Pr\left(p + \frac{3}{4}f' + \frac{Pr}{8}f^2\right)\bar{Y} = 2\,Pr\,f'p\frac{\partial \bar{Y}}{\partial p} \qquad (43)$$

with

$$\bar{Y}'(0) = -1, \qquad \bar{Y}(\infty) = 0. \qquad (44)$$

An appropriate series solution for $\bar{Y}$ is

$$\bar{Y} = \exp\{-[2\,Pr\,(p+\lambda)]^{\frac{1}{2}}\eta\} \sum_{n=0}^{\infty} u_n(\eta) \times [2\,Pr\,(p+\lambda)]^{-(n+1)/2} \qquad (45)$$

in which $(p + \lambda) > 0$, $u_0(0) = 1$ and $u_n'(0) = u_{n+1}(0)$ for $n \geqslant 0$. The subsequent *modus operandi* is analogous to the case for which the wall temperature undergoes a step change. Thus, we shall merely present the results and supplement them with comments when desirable.

In (45), the functions u_n were found to satisfy the recurrence relationships (23) and (24) provided that, in the latter, the last term is replaced by

$$-(n-2)\,Pr\,f'u_{n-3}.$$

Also,

$$\left.\begin{aligned}
&u_0(0) = 1, \quad u_1(0) = 0, \quad u_2(0) = \lambda\,Pr, \quad u_3(0) = \frac{1}{8}a\,Pr, \quad u_4(0) = \frac{3}{2}\lambda^2\,Pr^2,\\
&u_5(0) = \frac{1}{2}a\lambda\,Pr^2, \quad u_6(0) = -\frac{1}{16}a^2\,Pr + \frac{67}{128}a^2\,Pr^2 + \frac{5}{2}\lambda^3\,Pr^3,\\
&u_7(0) = \frac{3}{2}a\lambda^2\,Pr^3, \quad u_8(0) = -\frac{7}{16}a^2\lambda\,Pr^2 + \frac{469}{128}a^2\lambda\,Pr^3 + \frac{35}{8}\lambda^4\,Pr^4,\\
&u_9(0) = \frac{77}{512}a^3\,Pr - \frac{871}{512}a^3\,Pr^2 + \frac{3297}{512}a^3\,Pr^3 + 4a\lambda^3\,Pr^4, \text{ etc.}
\end{aligned}\right\} \qquad (46)$$

Since

$$\bar{\theta}(0) = p^{-1}\bar{Y}(0) = p^{-1}\sum_{n=0}^{\infty} u_n(0) \times [2\,Pr\,(p+\lambda)]^{-(n+1)/2},$$

the dimensionless wall temperature is given by

$$\theta_w \equiv \theta(0,\tau) = \sum_{n=0}^{\infty} \frac{u_n(0)}{(2\lambda\,Pr)^{(n+1)/2}} \times \frac{\Gamma_{\lambda\tau}[(n+1)/2]}{\Gamma[(n+1)/2]} \qquad (47)$$

As $\tau \to \infty$, (47) becomes

$$\theta_{w,s} \equiv \theta_s(0) = \sum_{n=0}^{\infty} \frac{u_n(0)}{(2\lambda\, Pr)^{(n+1)/2}} \tag{48}$$

from which λ can be determined as shown in the following section.

(B.1) *Steady-state solution and evaluation of λ*

An inspection of (12b) and (13b) shows that the steady-state temperature field $\theta_s(\eta)$ satisfies

$$\theta_s'' + Pr\, f\theta_s' = Pr\, f'\theta_s \tag{49}$$

with

$$\theta_s'(0) = -1, \qquad \theta_s(\infty) = 0. \tag{50}$$

The power series solution for θ_s is

$$\theta_s = b - \eta + \sum_{n=2}^{\infty} \frac{\theta_s^{(n)}(0)}{n!}\eta^n \tag{51}$$

where,

$$b \equiv \theta_s(0) \quad \text{and} \quad \theta_s^{(2)}(0) = 0, \quad \theta_s^{(3)}(0) = ab\, Pr, \quad \theta_s^{(4)}(0) = -a\, Pr,$$

$$\theta_s^{(5)}(0) = 0, \quad \theta_s^{(6)}(0) = -a^2 b\, Pr(1 + 2\, Pr), \quad \theta_s^{(7)}(0) = a^2\, Pr(4 + 5\, Pr),$$

$$\theta_s^{(8)}(0) = 0, \quad \theta_s^{(9)}(0) = a^3 b\, Pr(11 + 28\, Pr^2), \quad \text{etc.}$$

The unknown parameter b has yet to be determined. To this end, we formally integrate (49), regarding it as a nonhomogeneous equation in θ_s', to obtain

$$\theta_s' = -\exp(-Pr\, F) + Pr \exp(-Pr\, F)\int_0^\eta f'\theta_s \exp(Pr\, F)\, d\eta \tag{52}$$

where $F = \int_0^\eta f\, d\eta$. This operation is motivated from the physical consideration that θ_s' is a rapidly changing function with η in the boundary layer. A second integration from $\eta = 0$ to $\eta = \infty$ yields

$$-b = -\int_0^\infty \exp(-Pr\, F)\, d\eta + Pr\int_0^\infty \exp(-Pr\, F)\, d\eta \int_0^\eta f'\theta_s \exp(Pr\, F)\, d\eta. \tag{53}$$

We recognize that the first integral in (53) equals $C(Pr)/0{\cdot}4790\, Pr^{\frac{1}{3}}$. To evaluate the double integral, we first transform the variable of integration from η to the Meksyn variable $F(= \int_0^\eta f\, d\eta)$ using the series inversion techniques, followed by expanding the exponential in the inner integral, integrating and evaluating the outer integral in gamma functions. The result is

$$-b = -\frac{C(Pr)}{0{\cdot}4790\, Pr^{\frac{1}{3}}} + \left[b - \frac{2}{9}\left(\frac{6}{a}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{Pr^{\frac{1}{3}}} + 0 + \frac{4}{5}b + \frac{1}{405}\left(\frac{1}{Pr} - 75\right)\left(\frac{6}{a}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{Pr^{\frac{1}{3}}} \right.$$

$$+ 0 - \frac{b}{40}\left(\frac{1}{Pr^2} + \frac{17}{5\, Pr} - 30\right) - \frac{4}{3645}\left(\frac{1}{Pr^2} - \frac{33}{2\, Pr} + 150\right)\left(\frac{6}{a}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{Pr^{\frac{1}{3}}} + 0 - \frac{3b}{2200}$$

$$\left. \times \left(\frac{1}{Pr^3} + \frac{21}{Pr^2} + \frac{130}{Pr} - \frac{1640}{3}\right) + \ldots \right]. \tag{54}$$

For an assigned value of Pr, the corresponding b can be determined from (54). However, the series in (54) is, in general, semi-divergent; Euler's transformation was first applied followed by a determination of the sum according to Shanks [9]. This was done since the resulting series after the Euler's transformation was slowly convergent. Such procedure has been found to

Table 2. Values of b and λ

Pr	0·01	0·1	0·72	1	10	100
$b[=\theta_s(0)]$	6·876	3·556	1·720	1·536	0·7071	0·3277
λ	4·735	1·909	1·132	1·022	0·4815	0·2241

Table 3. Comparison of results for $\theta_s(0)$

Pr	0·7	1·0	2·0	10	20	100
from (54)	1·737	1·536	1·212	0·7071	0·5606	0·3277
Levy [10]	1·758		1·233	0·7262	0·5811	
Sparrow and Lin [11]	1·742	1·541		0·7086		0·3286

be particularly useful when only a limited number of terms is available. For the smallest Pr (0·01) considered, the resulting series of the sums did not converge; Shanks' e_2 transformation was applied at the outset as the original series contained zeros. Having evaluated the b's, the corresponding λ's were calculated from (48) in a manner identical to that used in case (A). The numerical values of b and λ so determined are summarized in Table 2.

Levy [10] reported results of a numerical computation for wall temperature gradients in laminar boundary layers in wedge flow with power law wall temperature distribution and for Pr ranging from 0·70 to 20. Information on wall temperature of a flat plate with uniform heat flux can be readily deduced from his data. Sparrow and Lin [11] conducted an analysis of wall temperature variation for flow over a flat plate with power law heat flux distribution, including the uniform wall flux as a special case. Prandtl numbers of 0·7, 1, 10 and 100 were considered. Like [10], straight numerical computation was used in [11]. Table 3 compares the results of our calculation with those evaluated from [10] and [11]. It is seen that the agreement is very good, particularly with the data of Sparrow and Lin.

The result of prime interest here is the wall temperature response subsequent to the step change of wall flux. It is given by (47) and is valid for all times. As in case (A), expressions useful for small and large times can be readily deduced. They are listed below.

(a) *Solution useful for small time.*

$$\theta(0,\tau) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} Pr^{-\frac{1}{2}}\tau^{\frac{1}{2}} + \frac{a}{64}Pr^{-1}\tau^{2} - \frac{a^2}{840\sqrt{(2\pi)}}Pr^{-\frac{3}{2}}\left(1-\frac{67}{8}Pr\right)\tau^{\frac{7}{2}} + \frac{a^3}{1\,968\,080}Pr^{-4}(77 - 871\,Pr + 3297\,Pr^2)\tau^5 + \qquad (55)$$

which, upon inserting the numerical value of $a\,(=0{\cdot}4696)$, becomes

$$\theta(0,\tau) = 0{\cdot}7979\,Pr^{-\frac{1}{2}}\tau^{\frac{1}{2}} + 0{\cdot}7338\times10^{-2}Pr^{-1}\tau^2 - 0{\cdot}1047\times10^{-3}Pr^{-\frac{3}{2}}(1-8{\cdot}375\,Pr)\tau^{\frac{7}{2}} + 0{\cdot}405\times10^{-5}Pr^{-4}(1-11{\cdot}3\,Pr+42{\cdot}8\,Pr^2)\tau^5+\ldots \qquad (56)$$

Again, we note that λ does not appear in (55) or (56). The authors are not aware of any prior published analysis for the problem and, thus, no comparison can be made.

(b) *Asymptotic series useful for large times.*

$$\theta(0,\tau) = \theta_s(0) - e^{-\lambda\tau}\sum_{n=0}^{\infty}\frac{u_n(0)}{(2\lambda\,Pr)^{(n+1)/2}}\frac{(\lambda\tau)^{(n-1)/2}}{\Gamma[(n+1)/2]}\left[1+\left(\frac{n-1}{2}\right)\frac{1}{\lambda\tau}+\left(\frac{n-1}{2}\right)\times\left(\frac{n-3}{2}\right)\frac{1}{(\lambda\tau)^2}+\ldots\right] \qquad (57)$$

The results in this section are displayed graphically in Fig. 4 by plotting the wall temperature ratio $\theta_w/\theta_{w,s}[=\theta(0,\tau)/\theta_s(0)]$ vs. the dimensionless time τ for Pr ranging from 0·01 to 100. If one replots the wall temperature ratios against $Pr^{-\frac{1}{3}}\tau$, the data could be brought to lie on a single curve for Pr ranging from 0·72 to 100. This is illustrated in Fig. 5. Hence, for fluids with Prandtl number falling in the said range, the response time of the thermal layer

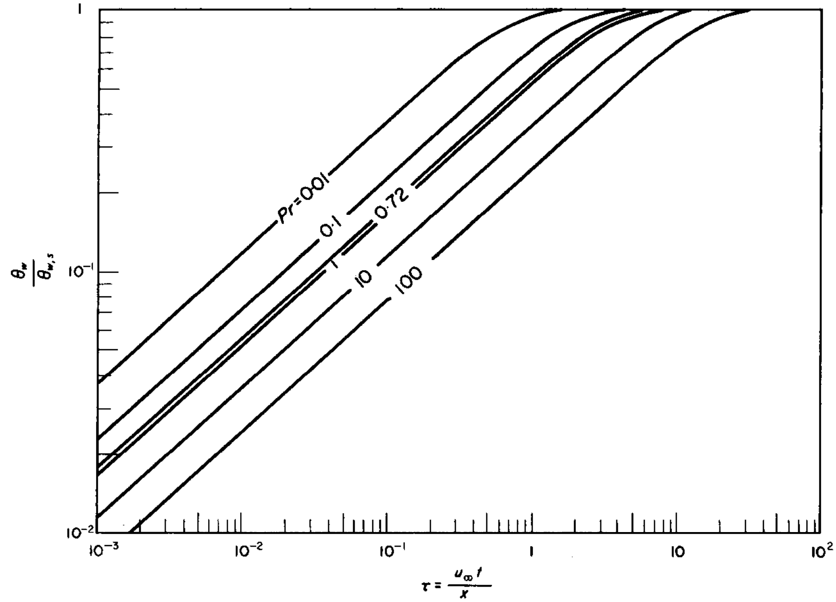

FIG. 4. Wall temperature response of a flat plate due to a step change in surface heat flux.

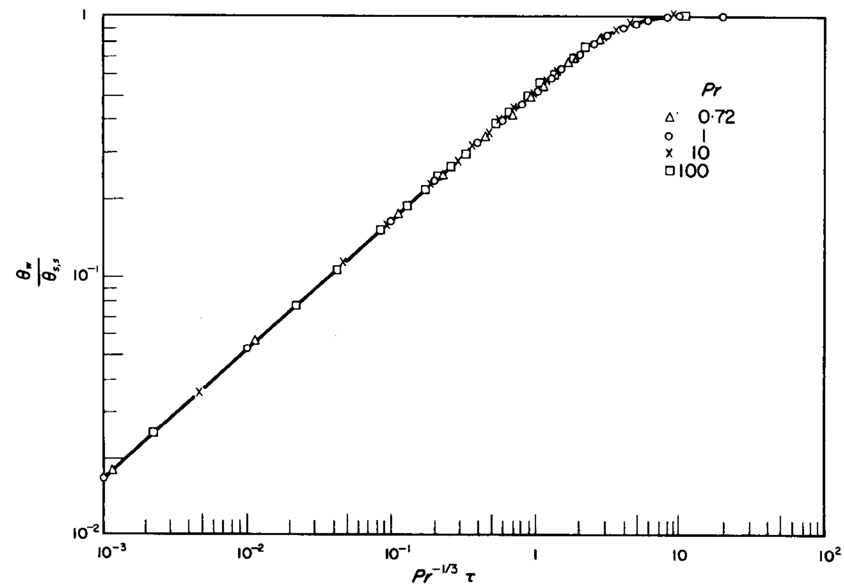

FIG. 5. Correlation of data used in plotting Fig. 4 for Pr = 0·72, 1, 10 and 100.

due to a step change in wall flux again varies directly as $Pr^{\frac{1}{3}}$.

4. COMPRESSIBLE FLOW

It is well known that for compressible flows, the momentum and the energy equation are coupled. A disturbance in the temperature field would induce a disturbance in the velocity field even though u_∞ is kept constant. However, under certain restrictive conditions, the incompressible results obtained in the previous section can be directly used to describe the thermal response of compressible flows over a flat plate. In the latter case, the governing conservation equations are:

continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad (58)$$

momentum:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad (59)$$

energy:

$$c_p\rho\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y} \times \left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2. \qquad (60)$$

There will be an initial velocity and temperature field in the fluid corresponding to those of an adiabatic plate. The precise prescription of such fields would, in general, depend on the equation of state of the gas and the manner by which its viscosity, conductivity and specific heat vary with temperature.

In what follows, we consider only an ideal gas of constant Prandtl number with viscosity varying linearly with temperature according to:

$$\frac{\mu}{\mu_\infty} = A\frac{T}{T_\infty} \qquad (61)$$

wherein A is a constant. This relation was first proposed by Chapman and Rubesin [12] and found to be reasonably accurate for common gases within moderate temperature ranges. Following Moore [13], we introduce a stream function $\psi(x, y, t)$ defined by

$$u = \frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial y}, \qquad v = -\frac{\rho_\infty}{\rho}\left(\frac{\partial \psi}{\partial x} + \frac{\partial}{\partial t}\int_0^y \frac{\rho}{\rho_\infty}\,\mathrm{d}y\right) \qquad (62)$$

and a coordinate transformation

$$Z = \int_0^y \frac{\rho}{\rho_\infty}\,\mathrm{d}y. \qquad (63)$$

It has been found that the momentum equation then becomes identical to that for the corresponding incompressible flow of a fluid with kinematic viscosity $A\nu_\infty$. Hence, the thermal effects on property variations are removed in the transformed x, Z space. Accordingly, upon introducing the similarity parameter η, redefined as

$$\eta = Z\left(\frac{u_\infty}{2A\nu_\infty x}\right)^{\frac{1}{2}} \qquad (64)$$

and a function $f(\eta)$ such that

$$\psi = (2A\nu_\infty u_\infty x)^{\frac{1}{2}} f, \qquad (65)$$

we may readily demonstrate that f is precisely the Blasius function and is given by (9).

In the transformed x, Z space, the energy equation takes the form:

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial Z}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial Z} = A\kappa_\infty\frac{\partial^2 T}{\partial Z^2} + \frac{A\nu_\infty}{c_p}\left(\frac{\partial^2 \psi}{\partial Z^2}\right)^2 \qquad (66)$$

which is recognized to be the same as that for an incompressible fluid of thermal diffusivity $A\kappa_\infty$. If we define a dimensionless time τ according to (10) and dimensionless temperatures θ according to (11a) for the case of step change in wall temperature and

$$\theta = \frac{T - T_\infty}{\frac{q_w}{k_\infty}\left(\frac{2v_\infty x}{Au_\infty}\right)^{\frac{1}{2}}} - \frac{u_\infty^2}{2c_p \frac{q_w}{k_\infty}\left(\frac{2v_\infty x}{Au_\infty}\right)^{\frac{1}{2}}}\Phi(\eta) \quad (67)$$

for the case of step wall flux change, we find that $\Phi(\eta)$ satisfies (14, 15) and θ satisfies (12a, 13a) and (12b, 13b) respectively for the two cases. Hence, for the former, all expressions for $\theta'(0, \tau)$ given in the preceding section remain valid. However, in (34) and (35), the relevant properties which occur in Nu and Re should be evaluated at the wall temperature, namely, $Nu = q_w x/(T_w - T_{aw}) k_w$ and $Re = u_\infty x/v_w$. For the case of step change in wall flux, all results given for $\theta(0, \tau)$ remain unaltered.

5. CONCLUDING REMARKS

The main idea of the solution technique explored in this paper, namely, the use of the new dependent variable $\bar{Y}$ and the introduction of the parameter λ in the series expansion, has wider application than the problem considered. The senior author has recently successfully employed the same technique in the study of the transient response behavior for the transport of heat and mass to a translating droplet with vigorous internal circulation. The parameter λ was found to depend on the Péclet number and the angular coordinate measured from the front stagnation. The results obtained show very good agreement with those evaluated from a separate, independent analysis.

The problem of unsteady, laminar, forced convection at the two-dimensional and axisymmetrical front stagnation has been treated by the senior author [14]. Two appropriate asymptotic solutions for small and large times were separately determined and joined. Undoubtedly, the problem can also be solved by the present technique which has the advantage that it not only leads to a solution valid for all times but also avoids the drawback of the lack of uniqueness in the large time approximation of the earlier method.

While the present analytical procedure provides a relatively simple and rapid means of calculating the instantaneous surface heat flux and temperature characteristics, it does not lend itself conveniently to the determination of the transient temperature field although, in principle, this can be done. A knowledge of such fields is useful in certain applications.

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Résumé—On étudie la réponse thermique d'une couche limite incompressible et laminaire sur une plaque plane semi-infinie due à une variation en échelon soit en température pariétale, soit en flux pariétal. Une nouvelle méthode analytique pourtrouver des solutions et valable pour tous les instants est présentée. Les résultats pour les transitoires de flux pariétal ou de température pariétale sont présentés graphiquement pour des nombres de Prandtl variant de 0,01 à 100. On montre que, dans une gamme restreinte de nombres de Prandtl (0,72 à 100), le temps nécessaire pour que la couche thermique devienne stationnaire varie comme l'inverse de la vitesse de l'écoulement libre et comme la puissance $\frac{1}{3}$ du nombre de Prandtl.

Pour un gaz idéal dont le nombre de Prandtl est constant et avec une relation linéaire viscosité–température du type Chapman–Rubesin, les résultats incompressibles dans les deux cas peuvent être directement appliquées lorsque les propriétés de fluide sont interprétées convenablement.

Zusammenfassung—Es wird das thermische Verhalten einer inkompressiblen, laminaren Grenzschicht untersucht an einer halbunendlichen ebenen Wand infolge einer stufenweisen Änderung entweder der Wandtemperatur oder des Wärmeflusses durch die Wand. Eine neue analytische Methode zur Auffindung von Lösungen für den ganzen Zeitraum wird angegeben. Ergebnisse für den Wärmestrom- oder Wandtemperaturverlauf sind graphisch für Prandtl-Zahlen von 0,01 bis 100 dargestellt. Es wird gezeigt, dass für einen beschränkten Bereich von Prandtl–Zahlen (0,72 bis 100) die zum Aufbau einer stationären thermischen Grenzschicht erforderliche Zeit sich umgekehrt mit der Freistromgeschwindigkeit und direkt mit der $\frac{1}{3}$ Potenz der Prandtl–Zahl ändert.

Für ein ideales Gas konstanter Prandtl-Zahl mit linearem Zähigkeits–Temperaturverhalten nach der Chapman–Rubesin-Art können die Ergebnisse für inkompressibles Verhalten für beide Fälle bei richtiger Auslegung der Stoffwerte direkt angewendet werden.